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Mathematics—Music of the Mind

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The Derivative of a Polynomial on Various Arcs of the Complex Domain

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Mathematical World News

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Reviews and Abstracts

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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Mathematics---Music of the Mind

The analogy is profoundly real. Only the open vision is needed to sense it. Music's mysterious magic lifts us to a far-away plane from which our ills, fears, doubts, discords and even our sorrows seem to merge harmoniously into a sound-picture that thrills, not pains the heart. So, through the ages since man's mind first began to reason on the causes of things seen but not understood, *Mathematics* has helped and guided that reasoning, harmonized its divergences, made rhythmic its confusions, brought into being, for its language, words and symbols of simple singleness of meaning. Those idea-melodies we call numbers must forever be acclaimed the alphabet of man's intelligence. For, the first lisping effort of reason in all men of every age and race was the effort to *Count*. It could not have been otherwise. Before all other questions, man has always asked and ever will ask: How many? How much? So, the melody of number must everlastingly run on—one, two, or three; four, five, or six;. . . .

But pathetic are the times that appear willing in effect to crucify the loftier reaches of this music of mind while honoring only the street organ number melodies that count the pennies of wealth, or gratify the greed of bargaining! Away with Euclid, Newton, Leibniz and all the host of immortals whose mathematics brings no wage and pays no dividend! ?

S. T. SANDERS.

Enumeration of the Rational Points Between 0 and 1

By EDWIN L. GODFREY
University of Indiana

One of the requirements of a proof of the enumerability of the set of rational points on a line between 0 and 1 is that there shall be given a definite, orderly arrangement of the rational points, omitting none, according to a stated rule. Given such an arrangment of the points, a one-to-one correspondence with the set of positive integers may be made, and the enumerability of the set of rational points be established.

Examination of such an arrangement of the points suggests the possibility of finding an analytical expression, or an algorithm, by means of which when any rational number p/q is given, a definite positive integer is uniquely determined; and conversely, given any positive integer N, a definite one of the rational numbers between 0 and 1 is uniquely determined.

The arrangement of the points usually given is as follows:

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \ldots,$$

wherein, of two fractions with different denominators, that having the lesser denominator precedes, and of two fractions with the same denominator, that having the lesser numerator precedes; p and q are relatively prime.

Apparently no simple division of the above arrangement into groups can be made affording a ready expression of the relation between any rational number p/q and its position N in the sequence. However, another rule of formation of a similar sequence is suggested by the scheme given by Huntington,* in which the rational numbers are arranged in a two-dimensional array, subsequently reading the table diagonally.

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \ldots$$

*E. V. Huntington, *The Continuum and other types of serial order*, (p. 35) Harvard University Press, 1917.

$$\frac{2}{3} \quad \frac{2}{5} \quad \frac{2}{7} \quad \frac{2}{9} \quad \frac{2}{11} \quad \frac{2}{13} \quad \cdots$$

$$\frac{3}{4} \quad \frac{3}{5} \quad \frac{3}{7} \quad \frac{3}{8} \quad \frac{3}{10} \quad \frac{3}{11} \quad \cdots$$

$$\frac{4}{5} \quad \frac{4}{7} \quad \frac{4}{9} \quad \frac{4}{11} \quad \frac{4}{13} \quad \frac{4}{15} \quad \cdots$$

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

It is readily seen that such an arrangement omits none of the rational numbers, for all those with numerator p will appear in the pth row, and in the pth row follow successively all q's which are relatively prime to p and larger than p.

Arranging the fractions now in a continuous sequence by reading this table diagonally and writing the corresponding positive integer beneath each fraction we have

$$\frac{p}{q}: \ \frac{1}{2}; \ \frac{1}{3}, \ \frac{2}{3}; \ \frac{1}{4}, \ \frac{2}{5}, \ \frac{3}{4};$$

$$\frac{1}{5}, \ \frac{2}{7}, \ \frac{3}{5}, \ \frac{4}{5}; \ \frac{1}{6}, \ \frac{2}{9}, \ \frac{3}{7}, \ \frac{4}{7}, \ \frac{5}{6}; \ \cdots$$

$$N: \ 1; \ 2, \ 3; \ 4, \ 5, \ 6;$$

$$7, \ 8, \ 9, \ 10; \ 11, \ 12, \ 13, \ 14, \ 15; \ \cdots$$

The sequence may be written down directly without reference to the the square array by observing that the numerators follow the sequence 1; 1, 2; 1, 2, 3; 1, 2, 3, 4; 1, 2, 3, 4, 5; 1, 2, 3, 4, 5, 6; $\cdots$ forming groups (separated by semi-colons in the sequence written above), each succeeding one of which contains one more element than the preceding. The denominators of any group of fractions are found by reference to the preceding group; e. g., the denominator of the fraction with numerator 3 in the fifth group is 7, because 7 is the next higher number after 5 (5 being the denominator associated with numerator 3 in the fourth group) prime to 3. In the next (sixth) group the numerator 3 will have 8 as the denominator, since 8 is the next higher number after 7 prime to 3. The denominators in the next group after 5/6 in the above sequence are therefore 7, 11, 8, 9, 7, 7.

Because of the regular increase of the number of fractions in each group it is readily possible to find the group corresponding to a given N, and we have only the problem of devising an algorithm to determine the position of the Nth element in the group. Let us consider first the general problem of expressing the relationship between N and i and j, where $E_{(N)}$ is the Nth element in a sequence formed by reading a square array of elements E_{ij} from the top down diagonally to the left.

$$\begin{array}{cccccc} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & \cdots \\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & \cdots \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & \cdots \\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & \cdots \\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

The resulting arrangement of E gives

$$\begin{array}{l} E = E_{11};\ E_{12},\ E_{21};\ E_{13},\ E_{22},\ E_{31};\ E_{14},\ E_{23},\ E_{32},\ E_{41};\ E_{15},\ \cdots \\ N = \ \ 1;\ \ \ 2,\ \ \ 3;\ \ \ 4,\ \ \ 5,\ \ \ 6;\ \ \ 7,\ \ \ 8,\ \ \ 9,\ 10;\ 11,\ \cdots \end{array}$$

The group number, g, is obviously the sum of the subscripts i and j minus 1. The number of the element within the group is i. Then N is equal to $i+n$, where n is the number of the last element of the group just preceding the one in which N is found, n being given by $\frac{1}{2}g(g-1)$, which is the sum of the first $g-1$ integers. For example, the N corresponding to E_{32} is found thus:

$$i=3,\ j=2,\ g=i+j-1=4,\ n=\tfrac{1}{2}g(g-1)=6,\ N=i+n=9,$$

which is verified by examination of the above sequence.

We see that the numerator p of any fraction p/q of the square array corresponds to i. The denominator q is the jth number prime to p, regarding $p+1$ as the first prime to p. The denominator q may be conveniently designated by the symbol $P_{p,j}$. It is evident that this number is unique for any given p and j, and that it is quite readily determined for values of j close to p. Even when j is much greater than p, the value of $P_{p,j}$ may be found with much less effort than would be required to write down all the fractions of the series up to p/q, perhaps by using factor tables.

The algorithm for finding N when p/q is given may therefore be summarized by the following formulae:

$$i=p,$$

$$P_{p,j}=q,$$

$$g=i+j-1,$$

$$n=\tfrac{1}{2}g(g-1),$$

$$N=i+n,$$

or N may be expressed entirely in terms of i and j as follows:

$$N=\tfrac{1}{2}(i+j)^2-\tfrac{1}{2}i-\tfrac{3}{2}j+1.$$

If N is given and the value of p/q is desired, we simply divide N into two parts i and n, making n as large as possible by assigning the largest integral value possible to g; from these values those of i, p, and q are easily found using the above formulae.

For illustration let p/q be 8/17 and the value of N be required. $i=p=8$; $P_{p,j}=P_{8,j}=q=17$; (17 is the jth number prime to 8 beginning with 9 as the first. These numbers are 9, 11, 13, 15, 17, 19, $\cdots$) whence $j=5$; $g=8+5-1=12$; $n=\tfrac{1}{2}(12)(11)=66$; $N=8+66=74$. Thus 8/17 is the 74th element in the sequence, or $E_{(74)}=8/17$.

Again, let N be 75 and the value of p/q be required. $n=\tfrac{1}{2}g(g-1)$ $=\tfrac{1}{2}(12)(11)=66$ (66 being the largest possible value for n less than 75 since $\tfrac{1}{2}(13)(12)=78$); $i=N-n=75-66=9=p$; $g=12$; $j=g-i+1$ $=12-9+1=4$; $P_{p,j}=P_{9,4}=14=q$. Thus $p/q=9/14$, which is the 75th element in the sequence, or $E_{(75)}=9/14$.

The Derivative of a Polynomial on Various Arcs of the Complex Domain

By W. E. SEWELL
Georgia School of Technology

1. *Introduction.* Let $P_n(z)$ be a polynomial of degree n in $z=x+iy$ and let $|P_n(z)|\leqq M$ on a set E with boundary C. Then* $|P_n'(z)|$† $\leqq Mk(E)f(n)$, where $k(E)$ is a constant depending only on E and $f(n)$ is a function of n. In particular if E is the segment $(-1, +1)$ of the axis of reals it is known‡ that $|P_n(z)|\leqq M$, $-1\leqq z\leqq +1$, implies $|P_n'(z)\sqrt{1-z^2}|\leqq Mn$, $-1\leqq z\leqq +1$. *Montel*§ has extended this result (with a multiplicative constant) to generalized derivatives.

In this note we will show that for C an epicycloid or hypocycloid we have

$$\left|P_n'(z)\sin\frac{a\phi}{2b}\right|\leqq\frac{Mn}{2b},\quad z \text{ on } C,$$

where a and b are the constants and ϕ is the parameter in the standard parametric representation of these curves; for C a cardioid whose polar equation is $r=a(1-\cos\Theta)$ we have $|P_n'(z)\sin\Theta/2|\leqq Mn/a$, z on C; and for C a rose curve whose polar equation is $r=a\sin m\Theta$ we have $|P_n'(z)|\leqq M(m+1)n/a$, z on or within C. It should be noted that these evaluations are valid irrespective of the position of the curve in the plane because a translation and rotation

$$z=e^{i\alpha}w+w_0,\quad \alpha \text{ real},\ w_0 \text{ constant}, \tag{1}$$

transforms the curve into the particular one considered above. Applying (1) to $P_n(z)$ we obtain a plolynomial $p_n(w)$ of degree n in w and

$$|P_n'(z)|=\left|p_n'(w)\frac{dw}{dz}\right|=|p_n'(w)|,\text{ since }\left|\frac{dw}{dz}\right|=1.$$

*W. E. Sewell: (1) *Generalized derivatives and approximation by polynominals*, Transactions of the American Mathematical Society, Vol. 41 (1937), pp. 84-123; (2) *On the modulus of the derivative of a polynomial*, Bulletin of the Society, Vol. 42 (1936), pp. 699-701.

†$P'_n(z)$ denotes the first derivative of $P_n(z)$ with respect to z.

‡S. Bernstein, *Leçons sur les propriétés extrémales et al meilleure approximation des fonctions d'une variable réele*, Paris, 1926; see pp. 38-44.

§P. Montel, *Sur les polynomes d'approximation*, Bulletin de la Société Mathématique de France, Vol. 46 (1919), pp. 151-196. See also W. E. Sewell (1), above footnote.

The method is essentially the same as that used by the author* in the investigation of this general problem for the ellipse.

2. *Epicycloid and hypocycloid.* Let C denote either of these curves, which can be represented parametrically by the equations

$$(2)\qquad \begin{aligned} x &= (a \pm b)\cos\phi \mp b\cos\frac{(a \pm b)}{b}\phi \\ y &= (a \pm b)\sin\phi - b\sin\frac{(a \pm b)}{b}\phi, \end{aligned}$$

where the upper signs belong to the epicycloid and the lower signs to the hypocycloid. Here $a \pm b$ is divisible by b or, what is the same thing, b is a factor of a. For z on C we have

$$(3)\qquad z = x + iy$$

$$z = (a \pm b)\cos\phi \mp b\cos\frac{(a \pm b)}{b}\phi + i\left[(a \pm b)\sin\phi - b\sin\frac{(a \pm b)}{b}\phi\right].$$

From (3) it follows that $P_n(z) = Q(\phi)$ for z on C, where $Q(\phi)$ is a trigonometric polynomial of order $[(a \pm b)/b]n$ in ϕ. By hypothesis $|P_n(z)| \leqq M$ and hence $|Q(\phi)| \leqq M$, consequently by a theorem of Bernstein† we know that

$$(4)\qquad |Q'(\phi)| \leqq M\frac{a \pm b}{b}n.$$

And since

$$(5)\qquad P_n'(z) = Q'(\phi)\frac{d\phi}{dz}$$

we will investigate the modulus of $d\phi/dz$. From (3) we have

$$\frac{dz}{d\phi} = (a \pm b)\left[-\sin\phi \pm \sin\frac{a \pm b}{b}\phi + i\left(\cos\phi - \cos\frac{a \pm b}{b}\phi\right)\right].$$

*W. E. Sewell, *On the polynomial derivative constant for an ellipse,* The American Mathematical Monthly, Vol. 44, pp. 577-578.

†S. Bernstein, op. cit., p. 39.

Hence

$$\left|\frac{dz}{d\phi}\right| = |a\pm b|\left[\sin^2\phi \mp 2\sin\phi\sin\frac{a\pm b}{b}\phi + \sin^2\frac{a\pm b}{b}\phi \right.$$
$$\left. + \cos^2\phi - 2\cos\phi\cos\frac{a\pm b}{b}\phi + \cos^2\frac{a\pm b}{b}\phi\right]^{1/2},$$

and by trigonometric reduction we obtain

$$\left|\frac{dz}{d\phi}\right| = 2|a\pm b|\left|\sin\frac{a\phi}{2b}\right|. \tag{6}$$

By applying (4) and (6) to equation (5) we have

$$\left|P_n'(z)\sin\frac{a\phi}{2b}\right| \leqq \frac{Mn}{2b}, \quad z \text{ on } C. \tag{7}$$

For C the hypocycloid of four cusps $b=a/4$ and inequality (7) becomes

$$|P_n'(z)\sin 2\phi| \leqq \frac{2Mn}{a}, \quad z \text{ on } C.$$

3. *Cardioid.* In polar coordinates the equation of the cardioid, which we shall call C, is $r=a(1-\cos\Theta)$, $a>0$. But $z=r(\cos\Theta+i\sin\Theta)$ and hence

$$z = a[\cos\Theta - \cos^2\Theta + i(\sin\Theta - \sin\Theta\cos\Theta)]. \tag{8}$$

Consequently for z on C we see from (8) that $P_n(z)=Q(\Theta)$, where $Q(\Theta)$ is a trigonometric polynomial of order $2n$ in Θ. Therefore as above $|Q'(\Theta)|\leqq M2n$. Also by (8)

$$\frac{dz}{d\Theta} = a[-\sin\Theta + \sin 2\Theta + i(\cos\Theta - \cos 2\Theta)],$$

$$\left|\frac{dz}{d\Theta}\right| = a[\sin^2\Theta - 2\sin\Theta\sin 2\Theta + \sin^2 2\Theta$$
$$+ \cos^2\Theta - 2\cos\Theta\cos 2\Theta + \cos^2 2\Theta]^{1/2},$$

$$\left|\frac{dz}{d\Theta}\right| = 2a\left|\sin\frac{\Theta}{2}\right|.$$

Consequently

$$|P_n'(z)| \leqq \frac{Mn}{a\sin\frac{\Theta}{2}},$$

or

$$\left| P_n'(z) \sin \frac{\Theta}{2} \right| \leqq \frac{Mn}{a}, \qquad z \text{ on } C.$$

4. *Roses.* For a rose curve C the equation in polar coordinates is either

$$(9) \qquad r_1 = a \cos m\Theta \qquad \text{or} \qquad (10) \quad r_2 = a \sin m\Theta.$$

Thus

$$z_1 = a \cos m\Theta(\cos \Theta + i \sin \Theta) \quad \text{or} \quad z_2 = a \sin m\Theta(\cos \Theta + i \sin \Theta).$$

Hence

$$\frac{dz_1}{d\Theta} = a[-m \cos \Theta \sin m\Theta - \cos m\Theta \sin \Theta + i(-m \sin m\Theta \sin \Theta + \cos m\Theta \cos \Theta)]$$

and

$$\left|\frac{dz_1}{d\Theta}\right| = a[m^2 \cos^2\Theta \sin^2 m\Theta + 2m \cos \Theta \sin m\Theta \sin \Theta \cos m\Theta + \cos^2 m\Theta \sin^2\Theta + m^2 \sin^2\Theta \sin^2 m\Theta - 2m \cos \Theta \sin m\Theta \sin \Theta \cos m\Theta + \cos^2 m\Theta \cos^2\Theta]^{1/2},$$

$$\left|\frac{dz_1}{d\Theta}\right| = a[(m^2-1)\sin^2 m\Theta + 1]^{1/2}.$$

As above we see that for z on C we have $P_n(z) = Q(\Theta)$, where $Q(\Theta)$ is a trigonometric polynomial of order $(m+1)n$ in Θ. Consequently $|Q'(\Theta)| \leqq M(m+1)n$. Thus we have for (9)

$$(11) \qquad \left| P_n'(z)\sqrt{(m^2-1)\sin^2 m\Theta + 1} \right| \leqq \frac{M(m+1)n}{a}, \quad z \text{ on } C.$$

Similarly for (10) we obtain

$$(12) \qquad \left| P_n'(z)\sqrt{(m^2-1)\cos^2 m\Theta + 1} \right| \leqq \frac{M(m+1)n}{a}, \quad z \text{ on } C..$$

It should be observed that

$$\sqrt{(m^2-1)\cos^2 m\Theta + 1} \geqq 1, \quad \sqrt{(m^2-1)\sin^2 m\Theta + 1} \geqq 1,$$

and thus inequalities (11) and (12) can be put in the form

$$|P_n'(z)| \leqq \frac{M(m+1)n}{a}, \quad z \text{ on or in } C,$$

by application of the principle of the maximum.

Lejeune Dirichlet
in his earlier days

Rebecca Dirichlet
his wife

After 1885

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

Notes on Lejeune Dirichlet

By G. WALDO DUNNINGTON
University of Illinois

Gustav Peter* Lejeune Dirichlet was born February 13, 1805, at Düren, a village in the industrial district of the Rhineland between Cologne and Aix-la-Chapelle, where his father was a postmaster. The father was a gentle, pleasant, and amiable man; the cultivated and intellectual mother, Elise Dirichlet, gave the talented boy the thorough rudiments of a good education. This French emigrant family was Catholic, poverty-stricken, and had eleven children. Young Dirichlet received early instruction in the elementary school and when this no longer sufficed, he was sent to a private school so as to be prepared to enter the "Gymnasium" later on, and here he received special instruction in Latin. In mathematics he was extremely precocious. Before the age of 12 he used to spend all his pocket-money for the purchase of mathematical books, which he studied every spare moment, especially in the evening. To those who remonstrated with him and claimed that he couldn't understand these books, he replied: "I read them until I understand them." His parents wanted him to become a merchant, but he showed such a strong dislike to this that they yielded and sent him in 1817 to the Bonn Gymnasium.

Dirichlet's industrious attitude, his good moral character, and pleasing, modest personality began to make friends of all who came in contact with him. History and mathematics were his special interests; he enjoyed the study of public affairs, particularly the French revolution. He conversed on such matters with an independence unusual in one so young, and expressed liberal views which were probably the result of his parents' training. They had as French subjects witnessed the stirring period; thus Dirichlet regarded the French revolu-

*There is doubt whether Gustav or Peter was the first name; he usually signed himself G. Lejeune Dirichlet.

tion as the origin of all free movements on the Continent. He loved social life and conversation about his favorite subjects, but had no use for games or other pleasures of youth, or anything rough and crude.

Dirichlet remained only two years at the Bonn Gymnasium and transferred then to the Jesuit Gymnasium at Cologne at the request of his parents. Here he enjoyed the instruction of G. S. Ohm in mathematics and progressed so rapidly that in 1821 (only 16 years old) he was ready for the university and returned home to discuss with his parents the choice of a profession. They had given up the idea of his becoming a merchant, but suggested that he take up law as a lucrative career. He replied that if they insisted he would be obedient, but that he would devote his nights to the study of mathematics. They were sensible and finally consented to his becoming a mathematician.

The parents had contacts with a number of families in Paris, and thus Dirichlet went there in May, 1822, in order to study his favorite subject. He attended lectures at the Collège de France and the Faculté des Sciences where his teachers were Lacroix, Biot, Hachette, and Francoeur. In addition he attempted to attend (as visitor) lectures at the École Polytechnique, but was unsuccessful, because the Prussian chargé d'affaires at Paris would not undertake to apply for permission from the French Minister without a special authorization from the Prussian Minister von Altenstein. A fine example of red tape! He now underwent privations, his early Paris life was very simple and modest as to board, room, and clothing; he enjoyed little social life. Nevertheless he always looked back on this period with intense pleasure because he began now to drink up knowledge.

Besides attending lectures Dirichlet started the intensive study of the best mathematical literature, especially Gauss' *Disquisitiones arithmeticae*. In fact this book had a much more important influence on him than his other Paris studies; he studied it incessantly for the remainder of his life. Dirichlet is said to have been not only the first person who completely understood this treatise, but also the first one who explained its methods to others and offered alternative proofs (often simpler and clearer) without sacrificing the desired rigor. His studies were interrupted only once by an attack of small-pox; his social intercourse was limited to several homes where he had letters of introduction and to several young Germans studying in Paris.

In the summer of 1823 a great change took place in Dirichlet's "retired" life. General Foy, an intellectual, broadly cultured man, leader of the opposition in the Chamber of Deputies and one of its most celebrated orators, was looking for a tutor in German language

and literature for his children. He had had a brilliant military career and his home was one of the social centers of Paris. Larchet de Chamont was a mutual friend of the Dirichlet family and Foy; he arranged an interview with the result that young Dirichlet immediately made a most favorable impression on Foy. He accepted the position at a salary such that he was no longer obliged to expect financial assistance from his parents and had leisure time for mathematical research. The Foys treated him like a member of the family, he felt happy in the position, especially because he was thus enabled to complete his social training or other matters relating to etiquette. The instruction of the children (the eldest a girl of eleven) did not prove difficult and Madame Foy renewed her acquaintance with German, in return helping Dirichlet with his French. The influence of the General on him was a very wholesome one, he met the notables of the French capital and heard discussion of the great political issues that led to the July Revolution of 1830.

Dirichlet now completed his first published paper: *Mémoire sur l'impossibilité de quelques équations indeterminées du cinquième degré*; it was the result of his study of Fermat's Theorem. This paper was read in the Paris Academy on June 11, 1825; Lacroix and Legendre reported so favorably on it in the session of June 18 that it was decided to print it in the memoirs of the Academy. Dirichlet's scientific reputation was made now; he came into close contact with the leading members of the Academy, particularly Fourier, Sturm, and Alexander von Humboldt. Fourier aroused his interest in mathematical physics; his later work on Fourier series and integrals can probably be traced to this early stimulation. Humboldt, that rare and incurable connoisseur of young genius, always desiring to be helpful, assured Dirichlet that he ought to return to Germany, and with his usual enthusiasm boasted that a fine university position in mathematics would be easy to obtain. Later all his own influence was necessary to make this remark only approximately true.

General Foy died in November, 1825, and Humboldt soon returned to Berlin; Dirichlet thus decided to return to Germany, and, sponsored by Humboldt, applied to the Minister von Altenstein for a position at a Prussian university. He went in the fall of 1826 to Düren to visit his parents and wait for the new job, as well as to work on a new paper. Even with the help of Gauss and Encke the best that could be done for him was a position as Privatdozent at the University of Breslau with a salary of 400 thalers per year. The University of Bonn (in his native district) had meanwhile given him an honorary Ph.D. Dirichlet traveled to Breslau via Göttingen and visited Gauss

on March 18, 1827; he wrote his mother that he found a warm welcome. Gauss received him in a cordial fashion and the personal impression he made on Dirichlet was far more favorable than the latter had expected.

The *venia docendi* in Breslau was obtained by giving a trial lecture and discussion before the faculty, as well as writing a dissertation and publicly defending it in Latin. Now Dirichlet had never learned to speak Latin; he gave his trial lecture on the irrationality of π, after which he received permission from the ministry to dispense with the public Latin disputation and to hand in his Latin dissertation later. Some of the Breslau professors were shocked at this! Dirichlet's teaching in Breslau was not successful, his topics were too advanced and his methods were poorly adapted to the students, moreoever his modest demeanor did not impress them sufficiently or put them in the proper awe of him. His predecessor (whose name is scarcely remembered today) had written a textbook on analytic geometry! Since Dirichlet had done nothing of the sort, he did not belong in the same class, according to local opinion. During the stay in Breslau Dirichlet wrote two papers on biquadratic residues. The second was in Latin and constituted his *Habilitationsschrift.* Bessel and Fourier were especially captivated by these two papers. Fourier ranked Dirichlet's work higher than that of Jacobi and Abel in elliptic functions and expressed a wish that Dirichlet might return to Paris, in order to occupy a leading chair in the Academy.

Humboldt secured Dirichlet's promotion to an assistant professorship at Breslau and began negotiations to get him to Berlin. A position in mathematics at the Berlin Military Academy became vacant about this time and Humboldt recommended Dirichlet to General von Radowitz and the Minister of War. The objection was raised that he was too young (aged 23) to become a teacher of officers. The Minister von Altenstein granted him a one year's leave of absence from Breslau so that he could begin teaching *ad interim* at the Military Academy. Dirichlet went to Berlin in the fall of 1828 and at first he enjoyed the social life with young military officers of his own age. He had learned to like such people at the home of General Foy and had made a study of military history, so that there was a bond of common interest outside of mathematics. Later on, when Dirichlet had a large circle of research students in advanced mathematics at the University of Berlin, his position with the military students became a burden to him. Thus the Military Academy, which first brought him to Berlin, was afterwards one of the principal causes of his going elsewhere. When Thibaut of Göttingen died in 1832 Gauss wanted to propose

Dirichlet for the position, but did not do so because he felt that he would be unwilling to leave Berlin.

Soon after his arrival in Berlin Dirichlet took steps necessary for being allowed to lecture at the university there. His position and title of professor at Breslau did not *ipso facto* confer on him the *venia docendi* at Berlin. Thus he had to become Privatdozent again and sent this application to the faculty. The other initial formalities were dispensed with, and he began to lecture, still using the title of Privatdozent before his name, and was not made assistant professor until 1831. Several months later he was made a full member of the Berlin Academy.

In the late fall of 1828 Humboldt introduced Dirichlet to the Mendelssohn family, whose home was an important social center in the Berlin of that day. He seems to have fallen in love at first sight with Rebecca (b. April 11, 1811), the daughter of Abraham Mendelssohn-Bartholdy. She was a granddaughter of the great rationalistic philosopher and litterateur of the 18th century, Moses Mendelssohn; a niece of Dorothea Schlegel, and a sister of the famous musician and composer Felix Mendelssohn-Bartholdy.* (Incidentally it may be mentioned here that her first cousin, Ottilie Mendelssohn, married the mathematician Eduard Kummer.) In a letter dated Christmas, 1828, Rebecca's sister Fanny describes their first meeting with Dirichlet:

"As a counterpart of (Eduard) Gans† I must mention Dirichlet, professor of mathematics, a very handsome and amiable man, as full of fun and spirits as a student, and very learned. Gans quarrels and fights with him like a schoolboy. The above and several other young people spent Christmas eve with us. A large *Baumkuchen* was given to Dirichlet (he is inordinately fond of it). It was dressed as a lady and made him a declaration of love, which gave occasion for a thousand jokes."

For sometime Dirichlet met with resistance from Rebecca's parents, who may have preferred some of the many other suitors for her hand. Perhaps it was due to differences of race and religion, or doubt as to the seriousness and steadfastness of his love. Finally they yielded, owing largely to the entreaties of Fanny and her artist-husband, Wilhelm Hensel. Dirichlet and Rebecca became engaged on November 5, 1831, and were married in May, 1832. They settled in the large Mendelssohn home at Leipziger-strasse 3 and here their

*Rebecca was less gifted musically than Felix and Fanny, but surpassed them in acuteness of understanding, bright intellect and sparkling wit.

†Gans was in love with Rebecca and gave her lessons reading Plato in the original.

first child, Walter, was born on July 2, 1833. On July 9 Felix wrote to his sister:

"Thanks and praise to God, and all happiness and blessing to you, dear Rebecca. Well done! I am very glad! To Dirichlet I do not at least send a written congratulation, as he could not even find it in his heart to write a line to me on this occasion. He might at least have put down: $2+1=3$." Dirichlet is said to have been one of the most lazy correspondents.

In the spring of 1835 Dirichlet and his wife, accompanied by her family, went to Cologne, where Felix was conducting the music festival; from there they traveled to Ostend, and closed the vacation trip at Bonn, returning to Berlin on October 14. He seems to have gotten along famously with his wife and her people. In June, 1836, Rebecca had to go to the watering place Franzensbad, for her health. She took Walter and was joined in August by Dirichlet, who had been detained in Berlin by his teaching duties. The baths did her good and Dirichlet enjoyed the afternoon coffee hour in the park, with its conversation followed by the long drives. They saw various members of the royalty, also Metternich, and Dirichlet had an interview with Chopin. From Franzensbad they went into Bohemia, and traveled as far as Gastein, but cholera in Italy prevented them from going there.

At Munich (September, 1836) they got word of the death of Dirichlet's only surviving sister. He left his wife and son in Nuremberg and hurried to his parents, being now the only survivor of the eleven. His mother survived even him and died in 1868 at the age of one hundred. Dirichlet, his wife and Walter returned to Berlin in October.

Grief again came to Dirichlet in November, 1838; his second son, the handsome thirteen month old Felix, died, and Rebecca was very ill for some time. Her artist brother-in-law, Hensel, made three pictures of the child, which proved a consolation to the mother. Dirichlet's mother went to Berlin at this time. The following summer, 1839, Dirichlet took a trip to Paris with the thought of settling there, but gave up the notion. While he was absent, Rebecca and her sister spent the summer at Heringsdorf, then a picturesque primitive bathing resort on the Baltic seacoast.

Dirichlet and his son Walter spent the Christmas holidays of 1842 in Leipzig with Felix Mendelssohn-Bartholdy. The latter reported Walter to be a model boy, talented in drawing, and with a good ear for music. Dirichlet's Berlin home was burglarized shortly after his return.

Dirichlet's work on the analysis of infinite series and definite integrals originated in his study of mathematical physics, particularly Fourier's theory of heat. His applications of analysis to the theory of numbers proved especially fruitful and have often been compared to Descartes' application of algebra to geometry. One of his well known papers was on the solution of the Pell equation by means of circular functions. He reached independently a number of the same conclusions which Gauss, Euler, Legendre and Jacobi had already attained. Dirichlet is said to have been the first to introduce regular lectures at German universities on the theory of numbers. He gave two simpler and clearer new proofs of Gauss' law of quadratic reciprocity, new proofs of the theorems on the number of factorizations of numbers into three and four squares, and the general reduction of positive quadratic forms with three variables. From his pen there appeared only one paper on the theory of heat, two on the theory of potential (the second furnishing a new definition of the potential) and one on equilibrium in hydrodynamics. This last paper assumed that the fluid has originally the form of an ellipsoid.

Jacobi became acquainted with Dirichlet in 1829 when the former went from Königsberg to Berlin to visit relatives and friends there. They traveled together from there via Halle where they were joined by Wilhelm Weber (the physicist) on a trip to Thuringia. The two became close personal and scientific friends. Jacobi was ordered by physicians to spend the winter of 1843 in Rome, where C. W. Borchardt (the editor of Crelle's Journal) and Steiner were also spending some time. Dirichlet, although he seems to have had a certain preliminary aversion to Italy, had for some years been considering a trip there, and decided to use this opportunity.

In July, 1843, Rebecca and the two Dirichlet sons, Walter and Ernst, accompanied by a man servant and a maid, started for Italy, proceeding via Darmstadt, Heidelberg. Karlsruhe, Freiburg i. B. and Badenweiler, where they were to wait for Dirichlet who was detained by his teaching. Jacobi, Borchardt and Dirichlet overtook them in August at Freiburg, the former two going on ahead. Little Ernst was taken sick the last of August at Vevay, Switzerland, and the party was delayed there a few days. On the way Dirichlet grew a beard, much to the amusement of the others in the party.

The party did considerable mountain climbing on the way, enjoyed various wines, and Dirichlet managed to smuggle a box of cigars past the customs officers. Rebecca was troubled with seasickness on all the bays, lakes, etc. At Lake Como they caught up with Jacobi, who went on to Milan, where they followed him early in September.

Here Leonardo's "Head of Christ" had a profound effect on the three mathematicians. The party moved on to Genoa, the Gulf of Spezia, Carrara, Lucca, Modena and Florence. Jacobi was left behind and enjoyed himself in Pisa. Rebecca made this interesting comment about him: "To my mind he resembles Italy a little, for you have a good deal to get over before you reach the fine qualities; his mind, however, is a superior one in every respect." In another place she mentions his persuasive sarcasm. Speaking of Galileo's monument in Florence, she remarked: "It is disgusting to see how they idolize Galileo now, to make people forget their old sins, but if another were to arise among them today they would treat him quite as badly. It was very touching to me to see his garden at Bellosguardo, where he lived, in which he used to dig, as they forbade him even to talk about science."

Art held a special attraction for these tourists, Dirichlet's favorite painter having been Perugino and Jacobi always preferring "Annunciations". Leaving Florence the Dirichlets went via Incisa, Perugia, Assisi, Terni, Foligno, Spoleto, and Sette Vene to Rome. They arrived there on November 1, 1843 and took an apartment at 45 Via Capo le Case. At Florence they had had a special tutor, Signor Paperini for Italian lessons and the entire family made excellent progress. In Rome Dirichlet spent much time reading Boccaccio and studied Italian with his usual perseverance; Jacobi claimed that he "flogged his teachers till he made them teach him something, and every person he met he considered a teacher." The Dirichlets soon made friends with Prince Baldassare Boncompagni (1821-1894), longtime editor (1868-1887) of the first journal on the history of mathematics; his family was reputed to very stingy, they found.

Jacobi and Dirichlet visited Lady Somerville in Rome; she had never heard of the former, but knew something of his famous brother who had sent her a medal gilded by his galvanoplastic process. She talked of nothing but 'monsieur votre frère', which wounded his vanity. His anger upon returning to his home made him witty and amusing, so that he kept the Dirichlet circle laughing an entire evening. Dirichlet and Jacobi had a half hour's audience with Pope Gregory XVI, who talked about mathematics and mathematicians in Germany, and showed much more knowledge of the subject than Lady Somerville. They thought he had prepared himself beforehand. Dirichlet as a faithful Roman Catholic knelt and kissed the toe of his slipper, while Jacobi as a Jew, merely kissed his hand.

Berlin apartments at Leipziger Platz 18 were leased for the Dirichlets for their occupancy upon their return from Italy, but a series of

unexpected events intervened and delayed them for a number of months. Meanwhile they enjoyed the Easter music and services of Holy Week, the dances and masked balls of the Carnival Season. Rebecca reported that Dirichlet as a *philosophe retiré du monde* threw flowers and sugar-plums with a learned man's contempt for folly of all kinds." The two sons, Walter and Ernst, seem to have had a glorious time on the entire trip. Occasionally Dirichlet became so joyfully absorbed in his mathematical researches that Rebecca had to make him go out for a walk.

Early in May, 1844, the Dirichlets went to Naples and stopped at 31 Santa Lucia, where they enjoyed the view of Vesuvius. The weather however was so unfavorable and the city so noisy that they did not stay here long. Jacobi escorted them on board the *Ercolano,* as he was ready to return to Germany, and they proceeded to Palermo, Sicily, which they considered far superior to Naples, except for the fleas. Dirichlet stayed in Sicily longer than his family, so that he could visit Mt. Etna and the tomb of Archimedes. Rebecca waited for Dirichlet at Sorrento, having been taken ill in June. The sea-bathing and idyllic life at Sorrento made them reluctant to leave the place.

Rebecca's illness proved to be a severe case of "black jaundice" and in addition she was in the fifth month of pregnancy. The doctors in Rome "denied the possibility" of the latter condition (she herself thought it impossible) and treated her for jaundice alone, making matters worse. As a climax to this, Dirichlet had a violent attack of "Roman fever" and needed an immediate change of air. Ill as they were, they journeyed to Florence, where Dirichlet got worse. It was decided that due to his illness they must remain in Italy all winter. Jacobi volunteered to take over the main part of Dirichlet's work in the interim at the Military Academy and the University of Berlin, so as to avoid loss of salary. Rebecca continued her piano practice throughout the winter and waited for the blessed event. They had calculated on the beginning of April, but on February 13 (Dirichlet's own birthday) a healthy girl put in her appearance. They had desired a girl and named her Flora (sometimes called Florentina) from the fact of her having been born in Florence. She was christiened on March 12; Dirichlet left for Berlin early in April in order to resume his teaching, and Rebecca with the three children and an Italian nurse followed him on June 15. They visited at Freiburg i. B., Karlsbad, Mayence and Soden, arriving in Berlin on August 2, 1845.

Moritz Cantor once related that at Berlin during the summer semester of 1852 he was attending Dirichlet's course on definite inte-

grals. One day he was lecturing on one of his most famous discoveries, the theory of the discontinuity factor. At the close of his explanation he remarked: "That is quite a simple thought, but if one doesn't have it, one just doesn't have it", as he chuckled and stroked his beard. Borchardt said of him: "Dirichlet did not belong, like Euler and Jacobi, to those universal analysts, whose every thought embodies itself in a formula and every formula in turn gives rise to a thought, (also) not to those unterrified calculators who press forward through a labyrinth of formulas to the desired goal, he was rather one of those few mathematicians of whom it can be said that their entire activity is a production of thought." He preferred to clarify theorems on an intuitive or lower level, to analyze the foundations logically, and to avoid calculations as long as possible.

In 1846 the government of Baden tried unsuccessfully to get Dirichlet to the University of Heidelberg. When Gauss died in 1855 the University of Göttingen upon recommendation of Weber began negotiations to secure Dirichlet as his successor. His work at the Berlin Military Academy had become burdensome and he desired to be relieved of it, but wanted the university to pay the salary of this position in addition to his reular income. Moreover the ultra-liberal political views of himself and his wife played an important role now, since he was vigorously opposed to the reactionary measures adopted in Prussia at the time. He took advantage of this call to Göttingen and told the Prussian Ministry that he would accept the call, unless in the meantime his Berlin position was adjusted to his own satisfaction. The ministry hesitated and Wilhelm Weber took (in person) the formal Göttingen call to Dirichlet. Berlin now offered him more, but he felt bound by his previous statement and accepted, proud to be the successor of Gauss.

The family moved to Göttingen in the fall of 1855 and bought a large comfortable house with a pleasant yard at Mühlenstrasse 1; the home is marked today with a marble tablet. They were unusually happy there with scientific colleagues, with social friends, and soon became the center of a merry musical (and artistic) circle. Dirichlet had fewer students than in Berlin, but they were more to his liking; they were more gifted, better prepared, and more serious about their work; his audiences also included some of the younger instructors. Rebecca took into their home her nephew after his mother's death and raised him like one of her own children. Kronecker visited Dirichlet in the summer of 1856 in Göttingen and Dirichlet returned to Berlin for a visit in December, 1856. In the winter of 1857 Rebecca visited her relatives in Berlin, and in July, 1858, Dirichlet visited Kronecker,

who was summering at Ilsenburg. On the former's return trip to Göttingen Kronecker accompanied him by coach from Ilsenburg to Bad Harzburg. Later in the summer he went to Montreux on Lake Geneva for a rest and also to attempt completion of his paper on hydrodynamics, as well as to prepare for the Göttingen Royal Society of Sciences a eulogy of Gauss, to be delivered at the public fall session. This latter task was dropped and the work on hydrodynamics was never completed. He returned home with a serious heart ailment which had developed rapidly and alarmingly.

In the fall of 1858 Rebecca's nephew Sebastian Hensel and his bride visited the Dirichlets at Göttingen for four weeks; Dirichlet's aged mother is said to have been the life of the party. She was then ninety years of age but still manifested extraordinary energy, vitality, and a bright mind. "She joined in the most fatiguing mountain excursions, and was quite hurt if anybody ventured to offer her an arm in climbing, busied herself in the house and garden, and even took part in the dancing which was occasionally improvised in the evening."

Rebecca nursed Dirichlet devotedly and saw him well on the way to recovery, when on December 1, 1858, she herself died of a stroke, The blow was too great and Dirichlet followed her on May 5, 1859. His brain was removed and studied by the noted Göttingen physiologist, Rudolph Wagner, who did this in the case of a number of famous men and preserved their brains for comparative purposes in his collection. In the old St. Bartholomew Cemetery on Weender street in Göttingen, just several steps from the monument-adorned grave of the poet and ballad writer G. A. Bürger, lies Dirichlet, his grave surrounded by a low stone and brick fence. His successor was Riemann. Rebecca's brother Paul acted as a guardian and careful trustee for the orphan children of Dirichlet. The daughter Flora was taken into their home for several years. The late Göttingen professor of philosophy and apostle of Fries, Dr. Leonard Nelson (1882-1927), and the lawyer, Heinrich Nelson (d. 1928) of Walkemühle, were great-grandsons of Dirichlet and were the possessors of his papers.

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Dirichlet's personal character seems to have been genteel and amiable—model in almost every respect. Alexander von Humboldt described him as introspective. He was suspicious of philosophy because he said it had no unsolved problems (such as in mathematics) and thus did not realize its own limitations. He was modest to a fault and if living today would surely be accused of having an inferiority complex. Like Riemann, he did not like to speak or even appear in public. Rebecca records the fact that he never scolded her;

he had a habit of making low, polite bows to others. His conscientiousness and extreme caution were a byword among his friends.

The students of Dirichlet were especially devoted to him and he was never overbearing in dealing with them. Steiner, who had a special title for everybody, always addressed him as "Marquis". The great success of his teaching was due to the clearness and simplicity of his lectures, not to rhetorical eloquence or other artifice. Dirichlet was not a prolific writer; his work consisted in the main of short papers on some definitely limited topic. His style was distinguished by its rigor of proof and pure lucidity of method. His power of concentration was extraordinary; he could mathematicize on trips, hikes, at musicales or while mountain climbing. He is said to have solved a difficult problem in the theory of numbers while listening to Easter music in the Sistine Chapel at Rome. Dirichlet was extremely gregarious and, as the reader will have already concluded, very fond of travel. He hated to write letters, preferring to visit friends and discuss scientific or other interests with them. Thus his stay in Italy was mathematically most productive. Long mechanical calculations were an abomination to him.

Dirichlet is one of the great mathematicians who deserves more attention as well as becoming better known to most readers. Minkowski* said of him: "He possessed the art of uniting a maximum of keen-sighted thoughts with a minimum of blind formulas." He calls this "the true Dirichlet Principle". Dirichlet's influence is to be found in the work of Eisenstein, Kronecker, Riemann and Dedekind. He usually lectured for a small select circle of advanced students; in Göttingen Stern and Ulrich lectured for the larger groups. Dirichlet was never a member of an examination committee, and in Göttingen he took no part in the direction of the mathematics department.

*Minkowski has given the best account of Dirichlet's scientific work in a lecture delivered on the occasion of his Centenary, February 13, 1905, before the Göttingen Mathematical Society; to be found in Hermann Minkowski, *Gesammelte Abhandlungen*, B. G. Teubner, Leipzig & Berlin, 1911, Vol. 2, pp. 447-461: *P. G. L. Dirichlet und seine Bedeutung für die heutige Mathematik*; also printed in the Jahresbericht der Deutschen Mathematiker-Vereinigung, Vol. 14, pp. 149-163.

The Teachers' Department

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Mathematical Induction for Freshmen

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When a freshman is required to prove something by Mathematical Induction, in almost every case he does not have a clear conception of what he is expected to do. It is this absence of a knowledge of what is to be established that creates no small part of the difficulty in his study of Mathematical Induction. For instance in such a simple statement as the following:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2},$$

many students cannot give the meaning of the letter n. They have to be shown, or at any rate it is necessary to stimulate their thinking to get them to see that the letter n represents in this case, two things: (1) the ordinal, (2) the cardinal value of the term. In such a series as $4+8+12+\cdots+4n=2n(n+1)$ it is necessary to have them see that it is the symbol for only the positional or ordinal value of the term. Of course the cardinal value of the term is dependent upon its ordinal value.

Once the student attaches the correct meaning to the letter n, a more hopeful approach to the problem is possible. Hence a careful description of both members of the statement in any given example is most essential. To have him take time to select the words needed in this description is not time wasted. It is an excellent aid in arriving at an understanding of the problem. Let me illustrate by describing both members of the expression,

$$1+2+3+\cdots+n=\tfrac{1}{2}n(n+1).$$

This is the sum of the first n consecutive positive integers and equals one-half of the product of two consecutive positive integers, the smaller one being the number of terms in the series.

Again, take the statement $4+8+12+\cdots+4n=2n(n+1)$. This is the sum of the first n positive integers which are multiples of four and equals twice the product of two consecutive positive integers, the smaller one being the number of terms in the series. The description must reveal the meaning or meanings of the symbol n as well as that of any other letter or letters employed. These items, viz. the number of the term, the symbol for the value of any term, and the form of the right hand member in terms of the number of terms in the series are needed in setting up the problem to be proved. There should be a clear understanding of the problem. The description must be carefully given and just what is to be proved should be definitely stated.

One criticism to be made is that the pupil often thinks he has made a proof by merely making a substitution of $n+1$ for n. He has not. There must be some orderly, logical steps,—a process of reasoning which leaves no doubt as to what is desired or as to the accuracy of the method of its proof.

The arrangement which we present will consist of three parts. *First.* Assuming the statement true for a value n, where n represents any posit ve integer, and then showing by some devices or operations that the resulting statement is of the same form with respect to $n+1$ as the original is with respect to n, when n has been increased by unity. This part will be termed the auxiliary theorem. *Second.* Showing that the statement does hold true for a selected, positive integral value of n, such as 3, or 4 or 5, etc. *Third.* By a process of extension, showing that the formula holds true by induction for all positive integral values of n.

Let us illustrate by some examples, outlining the work in minute detail.

Show that $\qquad 1+2+3+\cdots+n=\frac{1}{2}n(n+1)$

First. (1) Description. This has already been done for this example earlier. Numbers (2) and (3) which follow will form the set-up of the problem, and the student must see just how (3) is formed from (2).

(2) Assume $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$

(3) To prove $1+2+3+\cdots+n+(n+1)=\frac{1}{2}(n+1)(n+1+1)$

Line (3) is formed from line (2) by analogy using the description. This makes the trans tion understandable, whereas to think of it as coming by substitution is misleading. The pupil is very apt to think that substitution is a part of the process or proof. But it plays no

part in the proof whatsoever. Substitution should be taboo, in the formation of (3). The student should convince himself that the series in (3) is formed from (2) by adding the next consecutive term to the left member according to the symbol for the value of the term. He should make sure also that the right member of (3) is of the same mould as that of (2) and in terms of the number of terms in the series. It is this analogously built form of the right member of (3) which demands a strictly logical proof. When he has achieved this understanding, he is ready to proceed with the proof.

PROOF

(4) If $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$ by assumption,

(5) since $n+1=n+1$ by suggestion and identity, (the suggestion is provoked by the added term on the left in (3)),

(6) then, $1+2+3+\cdots+n+n+1=\frac{1}{2}n(n+1)+(n+1)$ by equals added to equals,

(7) and $1+2+3+\cdots+n+n+1=\frac{1}{2}(n+1)(n+2)$ by combining and factoring.

We now state the auxiliary theorem. Since the right members of (7) and (3) are identical, it is evident that if the statement is true for a positive integral value of n, it is also true for the value $n+1$.

The pupil's power of observation and his recognition of the perfect correspondence, in form, and in terms of the number of terms in the series, of the right members of (7) and (2), are the outstanding things to be desired. When he has come to appreciate that some mathematical steps or operations are involved in passing from the assumed to the result of (7), he should be well on the road to an intelligent understanding of the process of Mathematical Induction.

Second. We now test the statement for a selected value of n, such as 4.

$$\text{Evidently } 1+2+3+4=\tfrac{1}{2}\cdot 4\cdot 5$$
$$\text{or } 10=10.$$

Thus it holds when $n=4$ by trial.

Third. We now employ Extension. Since it holds for $n=4$ by trial, it holds for $n=5$, by the auxiliary theorem. Again, since it holds for $n=5$, by proof, it holds for $n=6$, by the auxiliary theorem. This process may be extended indefinitely, thus establishing that the statement holds true for all positive integral values of n.

Second Example: Show that $2+2^2+2^3+\cdots+2^n=2(2^n-1)$.

First. (1) Description. This is the sum of the first n positive consecutive integral powers of 2 and is equal to twice the quantity, one less than "2 to the nth power", n being the number of terms in the series.

(2) Assume $2+2^2+2^3+\cdots+2^n=2(2^n-1)$.

(3) To prove $2+2^2+2^3+\cdots+2^n+2^{n+1}=2(2^{n+1}-1)$. (The student should form the right member of (3) by analogy and not by substitution.)

PROOF

(4) If $2+2^2+2^3+\cdots+2^n=2(2^n-1)$ by assumption,

(5) since $2^{n+1}=2^{n+1}$, by identity,

(6) then $2+2^2+2^3+\cdots+2^n+2^{n+1}=2(2^n-1)+2^{n+1}$, by "equals added to equals..."

(7) and $2+2^2+2^3+\cdots+2^n+2^{n+1}=2(2^{n+1}-1)$, by simplifying.

Auxiliary theorem. Since the right members of (3) and (7) are identical, it is evident that if the statement is true for a value n, it is also true for a value $n+1$.

Second. We now test the statement for a slected value of n such as 3.

Evidently $2+2^2+2^3=2(2^3-1)$

or $14=14$.

Third. We now employ Extension. Since it holds for $n=3$, by trial, it holds for $n=4$, by the auxiliary theorem. Again, since it holds for $n=4$, by proof, it also holds for $n=5$, by the auxiliary theorem. This process may be extended indefinitely, thus establishing that the statement holds for all positive integral values of n.

A Different Type of Example. Prove that x^n-y^n is always exactly divisible by $x-y$, if n is a positive integer.

First. (1) Description. This is the difference of the same positive integral powers of x and y, and x minus y is the difference of x and y. Here, n may be odd or even.

(2) Assume $x^n-y^n=(x-y)\cdot Q_1(x,y)$, where Q_1 is a function of x and y and is of degree $n-1$.

(3) To prove $x^{n+1}-y^{n+1}=(x-y)\cdot Q_2(x,y)$, where Q_2 is a function of x and y and is of degree n.

PROOF

Our assumption is that

(4) $x^n - y^n = (x-y) \cdot Q_1(x,y)$. Now we may write

(5) $x^{n+1} - y^{n+1} = x^{n+1} - xy^n + xy^n - y^{n+1}$ by identity.

This becomes, by factoring,

(6) $x^{n+1} - y^{n+1} = x(x^n - y^n) + y^n(x-y)$,

(7) and $x^{n+1} - y^{n+1} = x(x-y)\,Q_1(x,y) + y^n(x-y)$ by substitution of (4) in (6),

(8) and thus $x^{n+1} - y^{n+1} = (x-y)\,Q_2(x,y)$, where Q_2 is a function of x and y, and is of degree n.

(The question might be raised as to whether the Q_2 of (8) is the same as the Q_2 of (3), but that is somewhat irrelevant, since the problem is not to find the exact quotient in either case. The chief question is whether $x-y$ does exactly divide the difference of the higher powers or not.)

Auxiliary theorem. Since the right members of (8) and (3) are identical in form, at least, it is evident that if the statement is true for a value n, it is also true for a value $n+1$.

Second. We now test the statement for a selected value of n, such as 3. Evidently $x^3 - y^3 = (x-y)(x^2+xy+y^2)$.

Third. We now employ Extension. Since the statement holds true for $n=3$ by trial, it holds for $n=4$, by the auxiliary theorem. Again, since it holds for $n=4$, by proof, it also holds for $n=5$, by the auxiliary theorem. This process may be extended indefinitely, thus establishing that it holds for all positive, integral values if n.

These examples exhibit something of the variety of problems to which Mathematical Induction may be applied. It is only by insistence upon the continued application of this form of treatment or of some other form which another teacher may prefer, that the student will acquire an abiding understanding of a Mathematical Induction proof and its value. Some instructors may contend that this form of treatment is too formal or too prolix. But our contention is that such formality and detailed statement of the steps are the very types of work that a freshman in college needs, not only for his work in mathematics but for his other studies, particularly scientific studies.

Mathematical World News

Edited by
L. J. ADAMS

Professor Otto Spiess (Basel), in preparing the Bernoulli Correspondence for publication, reports that he is going through the papers of Maupertuis and La Condamine. Of the 300 Bernoulli letters only a few have been unearthed. He found to his surprise over one hundred Euler letters hitherto unknown. Photostats of them are being made in Basel. The University of Basel library has purchased the Gotha Collection of Documents, which contains over 5,000 letters both from and to the Bernoullis, a rich treasure-house of information concerning the 18th Century.

L. C. Hogben, Regius Professor of Natural History in the University of Aberdeen, is president of the London Branch of the English Mathematical Association. On Saturday, November 27, 1937, he delivered an address before the Association entitled, *The Needs and Difficulties of the Average Pupil.* The annual business meeting of this branch of the Association will be held on Saturday, January 29, 1938, at Bedford College, Regent's Park.

The Polish Academy of Sciences and Letters met at Cracow on June 14, 1937. Th. Banachiewicz read a paper on *The precision of an elliptical orbit determined from three observations.* S. Kaczmarz read a paper on *The Resolution of a System of Linear Equations by Successive Approximations.* These mathematical papers were in addition to the usual program of scientific research contributions.

Recent contributions of a mathematical nature to the National Academy of the Lincei, Rome, included:

G. Sansone. *Cesaro's Summability of the Laplace Series.*

R. Calapso. *Systems of lines of a surface which are invariant with respect to a transformation for a W congruence.*

The London Mathematical Society, through their publishers, C. F. Hodgson and Son, publish a manual entitled *Notes on the Preparation of Mathematical Papers.* It contains 19 pages and sells for 1 shilling.

At the April 22, 1937 meeting, G. N. Watson was appointed Editor of the Proceedings of the London Mathematical Society. Professor G. B. Jeffrey is president of the Society.

On the twelfth of September of this year Dr. Georg Hamel celebrated his 60th birthday. He began his scientific career as assistant to Felix Klein in Göttingen. At the present time he is on the editorial board of the *Zeitschrift fur angewandte Matematik und Mechanik.*

The program of the Mathematical Association of America, Twenty-second Annual meeting, Indianapolis, Indiana on December 30-31, 1937, included:

1. *Mathematical progress in theoretical economics.* Professor G. C. Evans, University of California.
2. *Stieltjes integrals.* Professor T. H. Hildebrandt, University of Michigan.
3. *The successive iterates of the Stieltjes transform expressed in terms of the elementary functions.* Professor D. V. Widder, Harvard University.
4. *A fundamental problem in the theory of numbers.* Professor H. W. Brinkmann, Swarthmore College.
5. *On convexity.* Professor L. L. Dines, Carnegie Institute of Technology.
6. *On a recent advance in statistical inference.* Professor H. L. Rietz, University of Iowa.

The National Council of Teachers of Mathematics held two sessions on Wednesday, December 29, at Indianapolis, Indiana. These sessions were scheduled to include:

1. *The need of cooperation between high school teachers and college teachers.* Professor A. J. Kempner, University of Colorado.
2. *What shall we do with these, our unfit?* Professor Joseph Seidlin, Alfred University.
3. *High school teaching values derived from the study of higher mathematics.* Professor J. O. Hassler, University of Oklahoma.
4. *Mathematics exhibits and the opportunities which they present.* Miss Laura E. Christman, Senn High School, Chicago.
5. *Omissions, enrichment and improved machinery needed in all three types of courses found in ninth grade mathematics.* Miss Edith L. Mossman, Berkeley, California.

6. *The role of demonstrative geometry in the cultivation of reflective thought.* Professor Harold Fawcett, Ohio State University.
7. *Geometry, a way of thinking.* Professor H. C. Christofferson, Miami University.

Ministerial Director Professor Dr. K. Th. Vahlen, Berlin, was elected an active member of the physical mathematical section of the Prussian Academy of Sciences.

The University of Wisconsin has appointed as instructors for this year: Dr. Churchill Eisenhart, Dr. Bernard Friedman, Dr. D. H. Hyers, Dr. S. B. Jackson, Dr. R. B. Kershner, and Dr. R. W. Wagner.

Professor C. G. Latimer of the University of Kentucky will be Visiting Lecturer at the University of Wisconsin for the second semester.

Problem Department

Edited by
ROBERT C. YATES

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, College Park, Maryland.

SOLUTIONS

Late Solutions: No. 161, *Karleton W. Crain*; No. 163, *C. W. Trigg* and *C. E. Springer*; No. 165, *C. W. Trigg*; No. 166, *Walter B. Clarke* and *C. W. Trigg*.

No. 167. Proposed by *Alfred Moessner*, Nurnberg-N, Germany.

What is the general solution of the system:

$$A+B+C+D=E+F+G$$

$$A^3+B^3+C^3+D^3=E^3+F^3+G^3$$

$$A^4+B^4+C^4+D^4=E^4+F^4+G^4$$

in integers?

Solution by *Johannes Mahrenholz*, Gottbus, Germany.

The identity:

$$(xt+yu)^2+(xu-yt)^2=(xt-yu)^2+(xu+yt)^2$$

establishes with the addition and subtraction of $(xu-yt)$ the further relation for $n=1,3$:

$$[(x-y)t+(x+y)u]^n+[2(xu-yt)]^n+[(x+y)(t-u)]^n+[2yt]^n$$
$$=[(x+y)t-(x-y)u]^n+[(x-y)(t+u)]^n+[2xu]^n.$$

If this expression presents a solution for $n=4$, it follows that the quantities x,y,t,u are the roots of the equation:

$$x^2u(u-3t)-2xyut+y^2(u^2+ut+2t^2)=0,$$

or

$$u^2(x^2+y^2)+ut[(x-y^2)-4x^2]+2y^2t^2=0.$$

One pair x,y determines two pairs u,t and reversibly. The following table lists sets of values satisfying the conditions:

x	y	u	t	A	B	C	D	E	F	G
1	0	3	1	4	6	−2	0	−2	4	6
7	3	3	1	17	18	−10	3	−1	8	21
7	3	3	29	73	−66	130	87	139	64	21
127	−42	3	29	2578	1599	1105	−1218	2704	979	381
127	−42	1176	617	204233	350532	−47515	−51828	−146299	303017	298704

No. 169. Proposed by *V. Thébault*, Le Mans, France.

Upon the sides AB, BC, CD, DA of a parallelogram $ABCD$ construct externally and internally the squares whose centers are E, F, G, H and E', F', G', H'.

(1) The quadrilaterals $EFGH$ and $E'F'G'H'$ are squares the sum of whose areas is equal to that of the squares constructed on two consecutive sides of the parallelogram.

(2) Construct the parallelogram $ABCD$ knowing the centers E, F, G, H (or E', F', G', H') of the squares constructed upon its sides.

(3) Determine the parallelogram $ABCD$ if each side of the square $E'F'G'H'$ passes through one of the vertices of the square $EFGH$.

Solution by *C. W. Trigg*, Cumnock College, Los Angeles.

(1) If isosceles right triangles be constructed internally and externally on the sides of the parallelogram as hypotenuses, the vertices of the right angles will be the centers of the squares in this problem. Set $AB=CD=a$, $BC=AD=b\geqq a$, $EF=c$, $G'F'=d$, and $\angle ABC=\phi$. Then $EB=CG=a/\sqrt{2}$, $BF=FC=b/\sqrt{2}$, and $\angle EBF=360°-90°-\phi$ $=270°-\phi$. $\angle FCG=180°-\phi+90°=270°-\phi$. So $\triangle EBF\cong\triangle FCG$ and $EF=FG$. Similarly it may be shown that $EF=FG=GH=HE$. Also $\angle EFB=\angle GFC$, and $\angle BFG=\angle BFG$, so $\angle EFG=\angle BFC=90°$. Hence, $EFGH$ is a square.

$BE'=CG'$, $BF'=CF'$, $\angle E'BF'=\phi-90°$, $\angle G'CF'=45°+45°-(180°-\phi)$ $=\phi-90°$. So $\triangle E'BF'\cong\triangle G'CF'$ and $E'F'=F'F'$. $\angle BF'E'=\angle CF'G'$,

$\angle G'F'B = \angle G'F'B$, so $\angle G'F'E' = \angle CF'B = 90°$. In like manner, $E'F' = G'F' = G'H' = H'E'$, and $E'F'G'H'$ is a square.

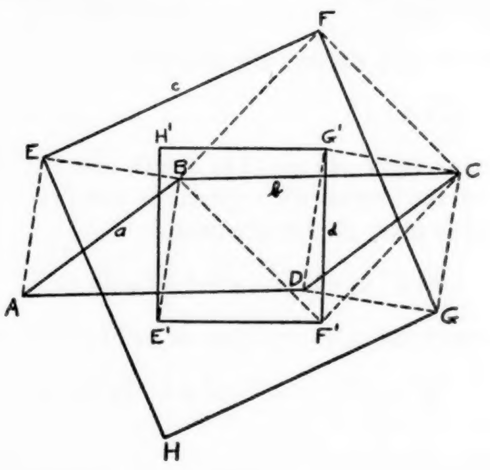
In $\triangle EBF$, $c^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - ab\cos(270° - \phi)$.

In $\triangle G'CF'$, $d^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - ab\cos(\phi - 90°)$.

Now since $\cos(270° - \phi) = -\cos(\phi - 90°)$, $c^2 + d^2 = a^2 + b^2$.

If $ABCD$ is a square, its vertices lie on the sides of $EFGH$ and $E'F'G'H'$ degenerates into the centroid of $ABCD$.

(2) As is evident from (1), if $EFGH$ (or $E'F'G'H'$) be given, and if congruent triangles be constructed on the exterior of two opposite sides and on the interior of the other two sides so that corresponding sides issue from the same point, the vertices of the obtuse vertex angles (or of the acute vertex angles) will be the vertices of the required parallelogram. Hence a unique solution will not be secured unless a fifth condition is imposed, e. g. one of the vertices of the other square



in position, a side or an angle of the parallelogram. (If the congruent triangles are right triangles, the vertices of the corresponding parallelogram are collinnear).

(3) As ϕ increases from 90° to 180°, c decreases from $\frac{1}{2}\sqrt{2}(a+b)$ to $\sqrt{(a^2+b^2/2}$, and d increases from $\frac{1}{2}\sqrt{2}(b-a)$ to $\sqrt{(a^2+b^2)/2}$. So the area of $EFGH$ is greater than the area of $E'F'G'H'$, hence the sides of the latter cannot pass through the vertices of the former, but the reverse is possible. When this occurs, $EE' = a$, falls along EH, and

$d^2=(c-a)^2+a^2$. In general, $a^2+b^2=c^2+d^2$ and $d^2=\frac{1}{2}a^2+\frac{1}{2}b^2-ab\cos(\phi-90°)=\frac{1}{2}a^2+\frac{1}{2}b^2-ab\sin\phi$. From these equations,

$$\sin\phi=\frac{\sqrt{2b^2-a^2}-a}{2b}$$

is the condition that the sides of *EFGH* pass through the vertices of *E'F'G'H'*.

Also solved by the *Proposer*.

Editor's Note. The following treatment may be of interest. Let the parallelogram be defined by the quantities ρt and 1, where ρ is the length and $t=e^{i\theta}$ the direction of one side. Let x_1 *represent* the centers of the squares (vertices of isosceles right triangles) external to the parallelogram. Then

$$x_1=(\rho/\sqrt{2})te^{i\pi/4} \qquad x_3=1+(\rho/\sqrt{2})te^{-i\pi/4}$$

$$x_2=\rho t+(1/\sqrt{2})e^{i\pi/4} \qquad x_4=(1/\sqrt{2})e^{-i\pi/4}$$

From these we may express the diagonals as:

$$x_1-x_3=i\rho t-1 \qquad x_2-x_4=\rho t+i.$$

It is clear that these are equal in length and at right angles since multiplication of the second by i produces the first. The square of the length of either gives double the area k:

$$2k=|x_2-x_4|^2=\rho^2+1+2\rho\sin\Theta.$$

On erecting interior triangles we have in like fashion for the area K':

$$2k'=|x'_2-x'_4|^2=\rho^2+1-2\rho\sin\Theta.$$

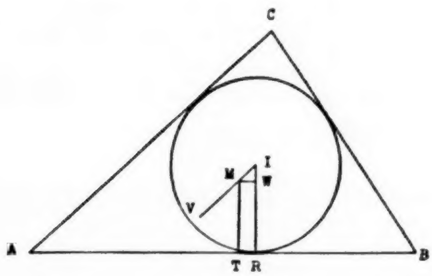
Thus:

$$\boxed{k+k'=\rho^2+1}$$

No. 172. Proposed by *Walter B. Clarke*, San Jose, Calif.

Show that if the X-line (the segment joining incenter to verbicenter) is parallel to one side of a triangle then the sides form an arithmetic progression, the constant difference being the length of the X-line.

Solution by *Karleton W. Crain*, Purdue University.



The equation of the X-line in trilinear coordinates is

$$\alpha(\sin C \sin A - \sin A \sin B) + \beta(\sin A \sin B - \sin B \sin C) + \gamma(\sin B \sin C - \sin C \sin A) = 0.^*$$

The condition that this be parallel to AC is,†

$$\begin{vmatrix} 0, & 1, & 0 \\ a, & b, & c \\ \left.\begin{matrix}\sin C \sin A \\ -\sin A \sin B,\end{matrix}\right\} & \left.\begin{matrix}\sin A \sin B \\ -\sin B \sin C,\end{matrix}\right\} & \left.\begin{matrix}\sin B \sin C \\ -\sin C \sin A\end{matrix}\right\} \end{vmatrix} = 0.$$

This may be written as,

$$\begin{vmatrix} 0, & 1, & 0 \\ a, & b, & c \\ ac-ab, & ab-bc, & bc-ca \end{vmatrix} = 0.$$

From this, $2b-a-c=0$, or $b=(a+c)/2$.

Therefore, a,b,c form an A. P.

Let M be the centroid, I the incenter, r the in-radius, V the verbicenter.‡

Let the perpendiculars to AB from M and I be TM and RI. Construct MW perpendicular to RI. Then, $\sin A = \sin \angle\, WMI = WI/MI = (RI-TM)/MI$. $MI = (RI-TM)/\sin A$. But $VI = 3MI$,‡ and $VI = 3(RI-TM)/\sin A$

$$= 3\left(r - \frac{b}{3}\sin A\right)/\sin A = (3r - b\sin A)/\sin A.$$

*Cf. solution of problem No. 156, N. M. M., November, 1937.
†See, for instance, Smith, Conic Sections, page 345.
‡Johnson uses the term Nagel point instead of verbicenter, see *Modern Geometry*, pp. 149, 184, 225.
§Johnson, *Modern Geometry*, page 225.

Since $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = 2sr/bc,$*

$VI = (3bc/2s) - b$. This reduces to $b-a$ if c is eliminated by means of the relation $2b-a-c=0$.

No. 173. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

Two variable spheres (X), (Y) cut a fixed plane along the same circle, real or imaginary. (A) and (B) are two concentric spheres, one orthogonal to (X), the other to (Y). The center X of (X) describes a fixed curve (F). Find the locus of the center Y of the sphere (Y).

Solution by the *Proposer.*

If a, b, are the radii of the two fixed spheres, and CL is the perpendicular from the common center C of these spheres upon the fixed radical plane of the two variable spheres, we have, both in magnitude and in sign,†

$$a^2-b^2=2LC\cdot XY.$$

Thus XY is fixed in magnitude, sense, and direction. Consequently the locus of Y is obtained from the locus (F) of X by a known translation.

No. 174. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

If the circumdiameters of the tetrahedron $ABCD$ passing through the vertices A, B, C, D meet the sphere again in the points P, Q, R, S, respectively, and the corresponding faces of $ABCD$ in the points K, L, M, N, we have

$$\frac{KP}{AK}+\frac{LQ}{BL}+\frac{MR}{CM}+\frac{NS}{AN}=2$$

Solution by the *Proposer.*

If O is the circumcenter of the tetrahedron $ABCD$, we have

$$\frac{KP}{AK}=\frac{OP-OK}{AK}=\frac{AO-OK}{AK}=\frac{AO}{AK}-\frac{OK}{AK}$$

*Cf. Passano, *Plane and Spherical Trig.*, (1918), pp. 74,75.

†Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 183, Art. 580. The Macmillan Company. 1935.

Now the sum of the ratios $AO:AK$ and its three analogues, and the sum of the ratio $OK:AK$ and its three analogues are respectively equal to three and one,* hence the proposition.

No. 175. Proposed by *G. W. Wishard*, Norwood, Ohio.

Show that any integer in an odd scale is odd, if and only if it has an odd number of odd digits. For example, 32400 in the quinary system is odd, because it has one odd digit. Again, 1357 in the nonary system is even, because it has an even number of odd digits, and is therefore divisible by 2.

Solution by *C. W. Trigg*, Cumnock College, Los Angeles.

Any integer in the scale of r, an odd base, may be represented as $a_0r^n+a_1r^{n-1}+\cdots+a_nr^0$. If a product has an even factor it is even, otherwise it is odd. Since r^k is odd, the characters of the members of the indicated sum are determined by the characters of the a's. Now the sum of even numbers is even, as is the sum of an even number of odd numbers. The sum of even numbers and an odd number of odd numbers is odd. Hence, an integer in an odd scale is odd, if and only if it has an odd number of odd digits.

Solution by the *Proposer*.

Let $2n+1=$any odd scale.

Reversing the order of the digits in the integer, we have

$$a+b(2n+1)+c(2n+1)^2+d(2n+1)^3+\&c\equiv$$

$$a+b+c+d+\&c \pmod 2.$$

Therefore, any integer in an odd scale is divisible by 2, if and only if the sum of its digits is divisible by 2.

But it is not necessary to add the digits.

For $1+2+3+4+5+6+7+8+9$

$$\equiv 1+0+1+0+1+0+1+0+1 \pmod 2$$

Any even digit $\equiv 0 \pmod 2$

Any odd digit $\equiv 1 \pmod 2$

The sum of any number of *even* digits $\equiv 0 \pmod 2$

*Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 115, Arts. 339 and 340, The Macmillan Company, 1935.

The sum of any *even* number of *odd* digits $\equiv 0 \pmod 2$

The sum of any *odd* number of *odd* digits $\equiv 1 \pmod 2$

Consequently, any integer in an odd scale is odd, if and only if it has an odd number of odd digits.

No. 179. Proposed by *V. Thébault*, Le Mans, France.

Find a perfect square of five digits such that the eight digits with which the number and its square root are written, in the system of base 8, are all different.

Solution by *C. W. Trigg*, Cumnock College, Los Angeles.

In the scale of 8, any number is congruent to the sum of its digits modulo 7. Also the sum of the eight distinct digits is congruent to zero modulo 7. So $N^2+N=N(N+1)\equiv 0 \pmod 7$, and N is of the form $7k$ or $7k-1$. Now $10234 \leq N^2 \leq 76543$, so $102 \leq N \leq 263$. Within this range there are but seventeen numbers of the required form which do not end in zero or one. Upon squaring these, the unique solution is found to be $(256)^2=73104$.

Also solved by *G. W. Wishard* who presents the following interesting octic squares, gathered from a table he has compiled for numbers up to 1000^2:

$66^2=5544$	$72^2=6444$
$106^2=11444$	$306^2=114444$
$265^2=77771$	$355^2=155551$
$526^2=344344$	$552^2=377744$
$603^2=444411$	$666^2=566544$
$732^2=666644$.	

These squares are easily checked by casting out the 7's.

PROPOSALS

No. 196. Proposed by *Walter B. Clarke*, San Jose, California.

The nine-point circle of a triangle is centered at the mid-point of the line joining incenter and circumcenter. Show that it bisects the three lines joining incenter to vertices.

No. 197. Proposed by *Walter B. Clarke*, San Jose, California.

Let I be the incenter of the triangle ABC, and let CI cut AB at D. If E and F lie on CA and CB so that angles

$$EDA = FDB = C/2,$$

will DE equal DF?

No. 198. Proposed by *Walter B. Clarke*, San Jose, California.

Given the triangle ABC with $a<b<c$. Let I be the incenter and M the mid-point of AB. D is taken on BC prolonged so that DM bisects the perimeter of the triangle. Let E be the contact point of the incircle with BC. Show that

$$CI : CE :: CA : DM.$$

No. 199. Proposed by *Walter B. Clarke*, San Jose, California.

Given the triangle ABC with $a<b<c$, and $B=60°$. Show that: (1) Side a plus intercepted length of the Euler line equals side c. (2) The sum of the three line segments from the orthocenter to the vertices is to the sum of the sides as the circumradius is to side b.

No. 200. Proposed by *Robert C. Yates*, University of Maryland.

Consider the directed triangle ABC.

(1) Give the construction of the circle through A and B which makes equal angles with AC and CB.
(2) Show that the three circles so constructed meet in a point.

No. 201. Proposed by *G. W. Wishard*, Norwood, Ohio.

Show that:

(1) Every perfect odd octonary cube ends with the same figure as the root.
(2) Every perfect even octonary cube ends with 0.
(3) Any number of zeros may be annexed to any perfect octonary cube, and the result will be a cube.

No. 202. Proposed by *V. Thébault*, Le Mans, France.

Consider six spheres (0_1), (0_2), (0_3), (0_4) and (S_1), (S_2). The radical planes of the spheres (S_1) and (S_2) taken successively with

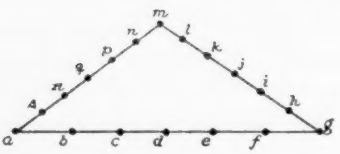
each of the spheres (0_i) determined by their intersections three by three, the vertices of two homologic tetrahedrons whose center and plane of homology are the radical center of the spheres (0_1), (0_2), (0_3), (0_4) and the radical plane of the spheres (S_1) and (S_2).

No. 203. Proposed by *V. Thébault*, Le Mans, France.

Spread the first 18 integers upon the perimeter of a triangle in such a fashion that

$$\begin{aligned}&a+b+c+d+e+f+g\\&\qquad=g+h+i+j+k+l+m\\&\qquad\qquad=m+n+p+q+r+s+a\end{aligned}$$

$$\begin{aligned}&a^2+b^2+c^2+d^2+e^2+f^2+g^2\\&\qquad=g^2+h^2+i^2+j^2+k^2+l^2+m^2\\&\qquad\qquad=m^2+n^2+p^2+q^2+r^2+s^2+a^2.\end{aligned}$$



No. 104. Proposed by *V. Thébault*, Le Mans, France.

In what system of numeration will there exist two perfect squares of four digits:

$$aa44=(cc)^2, \qquad 44aa=(dd)^2,$$

given that, in the decimal system, a is formed of two equal digits?

No. 205. Proposed by *A. Gloden*, Luxembourg.

What is the general solution in positve integers of the following diophantine systems:

$$\text{I.}\begin{cases}x+ay=u^2\\x^2+ay^2=v^2\end{cases}\qquad \text{II.}\begin{cases}x+ay=u^2\\x^2+bxy+ay^2=v^2\end{cases}$$

$$\text{III.}\begin{cases}x+y=u^2\\x^3+y^3=v^2\end{cases}\qquad \text{IV.}\begin{cases}x+y+z=u^2\\x^2+y^2+z^2=v^2\end{cases}$$

Reviews and Abstracts

Edited by
P. K. SMITH

Statistics in Education and Psychology. By Elmer R. Enlow. Prentice-Hall, Inc., New York, 1937. viii+180 pages. $2.75.

Dr. Enlow has arranged clearly and simply in one volume "the fundamental principles of statistical interpretation, patterns and the logic of mathematical computation, and all the tables ordinarily needed in statistical research." Each statistical measure has been covered with conciseness under the following subheads: Definition, Interpretation and Use, Formulas, Sampling Error (Reliability), Derivation of Formulas, Mathematical Properties, and Methods of Calculation.

The book contains chapters on frequency distributions, measures of central tendency and dispersion, probability and the normal curve, partial and multiple correlation, and errors in statistical work. In the appendices are given a bibliography, a glossary of symbols, and seven tables including alienation coefficients, Spearman R converted into Pearson r, and for the unit normal curve, abscissas and other values corresponding to assigned values of p and q, and areas and ordinates corresponding to given abscissas.

Although the book is designed as a textbook for class room use, we believe because of its logical presentation that it would serve as an excellent handbook to the occasional worker in statistics. The title words "in Education and Psychology" could be omitted, as statistics is treated quite generally. The publishers have presented the text in a very attractive form. The reviewer is impressed favorably.

Agnes Scott College. HENRY A. ROBINSON.

The Calculus. (Revised Edition). By Robert D. Carmichael, James H. Weaver and Lincoln Lapaz. Ginn and Company, Boston, 1937. ix+384 pages.

The authors state in the preface that the revision was undertaken with the cooperation of the departments of physics, mechanics and electrical engineering of the Ohio State University in order to meet the needs of engineering students. The usual order of presentation has been changed so that the calculus concepts may be used in technical

courses given simultaneously with the calculus. After the introductory chapter on functions and the theory of limits, the second chapter is devoted to derivatives (rational functions, sine and cosine, function of a function) and integrals (antiderivative, summation principle, definite integrals). General theorems on differentiation (including inverse functions, parametric representation, implicit functions) form Chapter III. Chapters IV to XI take up applications of differentiation and integration (with accent on mechanics), differential notation (arc length), maxima and minima of algebraic functions, differentiation of trigonometric functions and their inverse functions, differentiation of logarithmic and exponential functions (including four articles on hyperbolic functions), standard methods of integration (also improper integrals). Differential equations (variables separable, homogeneous, first and second order linear) are discussed in Chapter XII. The remaining chapters (XIII-XXII) treat successive differentiation and integration (theorem of the mean, undetermined forms), infinite series, power series expansions, properties of plane curves (concavity, inflection points, asymptotes, curvature), applications to geometry and mechanics, special methods of integration, functions of two or more variables partial derivatives, multiple integrals, geometric and mechanical applications). Each chapter is followed by a brief summary.

In general the text is well written. There is a good choice of problems likely to interest the technical student. In Chapter IV a careful distinction is made between velocity and speed; the distinction is all too often omitted in the usual calculus course. It is the reviewer's opinion that physical applications should be carefully and precisely discussed or else entirely omitted. It would have been desirable to to give more physical applications of differential equations. The only objections noted are the following:

Page 8, example 2. If $f(x)=\sqrt{1-x^2}$, then $f(\sin\Theta)=|\cos\Theta|$, not $\cos\Theta$

$$f(\cos\Theta)=|\sin\Theta|, \text{ not } \sin\Theta.$$

Page 23. In line with article 11 it would be better to write

$$\lim_{x\to\infty} |cx| = +\infty,\ c\neq 0 \text{ instead of } \lim_{x=\infty} cx=\infty,\ c\neq 0$$

$$\lim_{x\to 0} \left|\frac{c}{x}\right| = +\infty,\ c\neq 0 \quad \text{instead of} \quad \lim_{x=0} \frac{c}{x}=\infty,\ c\neq 0$$

The same remark applies to example 1, page 23. A question of notation appears also. The symbol $\lim\limits_{x=0} f(x)$ is somewhat misleading for frequently $x\to 0$ by values $x\neq 0$.

Page 43. Δy is a quantity susceptible of sign, hence it would be preferable to write $\Delta y = L\,Q$ instead of $Q\,L$.

Page 100. Length is essentially positive, so the formulas given for the tangent, normal, subtangent and subnormal do not always hold. Page 139. No accent on principal determinations of inverse trigonometric functions is made.

Page 159, example 9, states that

$$\tanh\frac{x}{2} = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}}$$

whereas, if x is negative a minus sign should appear before the radical.

Page 164. $\int \frac{dv}{v} = \log v$ should read $\int \frac{dv}{v} = \log |v|$

Similar remarks apply to several other fundamental formulas.

Page 228. In the statement of Rolle's theorem it is not necessary to require $f'(x)$ to be continuous. In many elementary problems $f'(x)$ may not be defined at the end points of the interval in question.

Page 340, example 5. In the problem of finding the envelope of

$$\frac{x}{a} + \frac{y}{b} = 1$$

with $a^2 + b^2 = k^2$, the value of b is given as

$$b = \sqrt{k^2 - a^2}.$$

In that manner half of the envelope is lost. This situation may be avoided by writing $a = k \cos \Theta$, $b = k \sin \Theta$.

The reviewer grants, however, that, were he to write a book on calculus, he would probably find many more objections to it than he does to this text.

This text is warmly recommended to the attention of all those mathematicians who have to prepare scientific students to use the calculus as a tool.

Virginia Military Institute. WILLIAM E. BYRNE.

Beiträge zur griechischen Logistik. By Kurt Vogel. (In the series *Sitzungsberichte der Bayerischen Akademie der Wissenschaften,* math.-naturwiss. Abteilung). Part I, C. H. Beck, Munich, 1936. Paper backs, 115 pp.

This fine piece of research by Dr. Vogel is dedicated to the memory of his late teacher, Heinrich Wieleitner, the noted historian. In the introduction he clearly shows the need for such a work based on the latest results of scholars, and states as a special purpose of this book the destruction of the old legend that we know almost nothing about practical Greek arithmetic, or logistics. It is proved here that it was sufficiently developed to meet the needs of daily life. Part I treats the fundamental operations with integers and fractions. Part II will contain the remaining sections of logistics and its applications (using much unpublished material), as well as a lexicon of the Greek terms involved.

In the historical introduction the evolution of logistics and arithmetic is studied, and it is found that a distinction occurs first in Plato's *Gorgias.* The logistic elements disappear almost completely in Euclid, but return in Archimedes. Number systems and methods of counting are treated in great detail in the next chapter, followed by a full discussion of each of the four fundamental operations with integers. In each case the Greek technical terms are always discussed carefully and collected.

A general survey of fraction calculation follows along with an investigation of the use of fractions in the various epochs of Greek mathematics. At the close of this work Vogel studies the *λόγος* (numerical ratio, for the comparison of two magnitudes) which often assumes the role of a fraction. It has been asserted that Greek mathematics did not have fractions, but that it always used these *λόγοι.* Vogel states that this is correct in that scientific mathematics tried to exclude fractions, but this was not entirely successful. The calculators could not free themselves from the concept of fractions. Some passages show that we cannot always interpret *ἀριθμός* as an integer. We are told that the idea of a fraction came into its own in the period of Archimedes, but that it was not clearly presented in the textbooks of the neo-Pythagoreans who inclined more to theoretical and philosophical arithmetic than to practical logistics; however, the idea of a fraction was not excluded before Euclid. In that time arithmetic and logistics were taught by the same teachers. In the end the exclusive use of *λόγοι* was confined to pure geometry.

Seven pages of important bibliographical items on Greek mathematics, and a nine page index form a useful conclusion to a task well

done. The typography, paper, etc., are attractive and misprints are scarcely to be found, with the exception of several in the index.

University of Illinois. G. WALDO DUNNINGTON.

Trigonometry. John W. Branson and J. O. Hassler. Henry Holt and Company, New York, 1937. 198 pages. Published with and without tables.

This is a good trigonometry. By a good trigonometry we mean here a text that is easy for the student to read, that is brief and concise and that stresses processes rather than rules. It is obvious that the authors have drawn upon the wisdom of years of experience in teaching the subject in order to anticipate students' difficulties in such matters as the meaning of the inverse function notation, explained with great clarity on page 109, and "peculiarity of logarithms of fractions", described on pages 24 and 25. Practical aids in computations are very helpful, as witness those on page 99 in finding the three angles of an oblique triangle, having given the three sides.

Other commendable features of this text include the historical notes, the interesting context of the problems and the particular attention to checking solutions.

The plan of the book merits consideration. The first chapter is entitled *Functions of Acute Angles,* and the solution of right triangles is introduced through the study of the tangent function, after which the other functions are defined and described. The other topics follow in this order: logarithms, functions of any angle, identities, introduction to oblique triangles, functions of more than one angle, oblique triangles concluded, inverse functions, graphs, complex numbers and spherical trigonometry. The appendix includes a review of algebra. It is interesting from an historical standpoint to note that the line value treatment of the functions has at last been relegated to the appendix.

Large printed italics are used to denote line segments on the figures, where possibly smaller letters would have made the diagrams clearer. No illustrative examples are given on changing radians to degrees, and *vice versa.* The tables are conspicuous for numerous omissions of decimal points, both in the entries for logarithms of trigonometric functions and in the accompanying tables of proportional parts. The index, as is customary with texts published both with and without tables, is between the answers and the tables in the latter edition, whereas it would be more convenient if placed after the tables.

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